Fundamental Concepts of Topology

Caleb Torres

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1 Introduction

The aim of this project is to demonstrate and verify several topological definitions and examples in order to convey a firm understanding of the most basic concepts in topology. The examples range across many different topological concepts, and reflect a necessary foundation for pursuing further coursework in topology.

2 Topological spaces, $(X, \mathcal{O})$.

Definition: A topological space is a set X together with a collection $\mathcal{O}$ of subsets of X called *open sets* such that:

1. The union of any collection of sets in $\mathcal{O}$ is in $\mathcal{O}$.
2. The intersection of any finite collection of sets in $\mathcal{O}$ is in $\mathcal{O}$.
3. Both $\emptyset$ and X are in $\mathcal{O}$.

The following examples give six topological spaces, $(X_i, \mathcal{O}_i)$, defined by a set X_i together with a topology $\mathcal{O}_i$.

1. $X_1 = \{a, b\}$ $\quad \mathcal{O}_1 = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$.
2. $X_2 = \{a, b, c\}$ $\quad \mathcal{O}_2 = \{X, \emptyset, \{b\}, \{c\}, \{a, b\}, \{b, c\}\}$.
3. $X_3 = \{a, b, c\}$ $\quad \mathcal{O}_3 = \{X, \emptyset\}$.

4. $X_4 = \{a, b, c, d\}$ $\quad \mathcal{O}_4 = \{X, \emptyset, \{a\}, \{a, b\}, \{a, c\}, \{a, b, c\}\}$.

5. $X_5 = \{a, b\}$ $\quad \mathcal{O}_5 = \{X, \emptyset, \{a\}\}$.

6. $X_6 = \{a, b, c, d, e, f\}$ $\quad \mathcal{O}_6 = \{X, \emptyset, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e, f\}\}$.

3 Open subsets of a topological space $(X, \mathcal{O})$.

A set U is open in a topological space $(X, \mathcal{O})$ if $U \subseteq \mathcal{O}$. The following examples give several open subsets U_i which are open on a specific topological space $(X, \mathcal{O})$.

For a topological space consisting of

$$X = \{a, b, c\} \quad \mathcal{O} = \{X, \emptyset, \{b\}, \{c\}, \{a, b\}, \{b, c\}\}.$$

let U_i denote the following open subsets in the topological space $(X, \mathcal{O})$ defined above.

$$U_1 = \{b\} \quad U_2 = \{c\}$$

$$U_3 = \{a, b\} \quad U_4 = \{b, c\}$$

$$U_5 = \{a, b, c\} \quad U_6 = \emptyset$$

4 Closed subsets of a topological space $(X, \mathcal{O})$.

Definition: A subset A of a topological space X is closed if its complement, $X - A$, is open.

The following examples give several closed subsets C_i which are subsets of a particular topological space $(X, \mathcal{O})$.

For a topological space consisting of

$$X = \{a, b, c\} \quad \mathcal{O} = \{X, \emptyset, \{b\}, \{c\}, \{a, b\}, \{b, c\}\}.$$

let C_i denote the following closed subsets in the topological space $(X, \mathcal{O})$ defined above.

$$C_1 = \{c\} \quad C_2 = \{a\}$$

$$C_3 = \{a, c\} \quad C_4 = \{a, b\}$$

$$C_5 = \emptyset \quad C_6 = \{a, b, c\}$$

5 Interior, closure, and boundary of $(X, \mathcal{O})$.

Given a subset A of a topological space X, then for each point $x \in X$ exactly one of the following holds:

1. There exists an open set O in X with $x \in O \subset A$.
2. There exists an open set O in X with $x \in O \subset X - A$.
3. Every open set O with $x \in O$ meets both A and $X - A$.

Points x where (1) hold form a subset of A called the interior of A or $\text{int}(A)$. The points x where (3) holds form a set called the boundary or frontier of A, denoted as ∂A. The points x where either (1) or (3) hold are the points x such that every open set O containing x meets A. Such points are called the limit points of A, and the set of these limit points is called the closure of A, written $\bar{A}$.

For a topological space consisting of

$$X = \{a, b, c, d, e, f\} \quad \mathcal{O} = \{X, \emptyset, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e, f\}\}.$$

let A_i denote any subset of X. Let $int(A_i)$ denote the interior of the set A_i, $\bar{A}_i$ the closure of the set A_i, and ∂A_i the boundary of the set A_i.

$$A_1 = \{a, b, c, d\}, \quad int(A_1) = \{a, c, d\}, \quad \bar{A}_1 = \{a, b, c, d, e, f\}, \quad \partial A_1 = \{b, e, f\}.$$

$$A_2 = \{a, c, f\}, \quad int(A_2) = \{a\}, \quad \bar{A}_2 = \{a, b, c, d, e, f\}, \quad \partial A_2 = \{b, c, d, e, f\}.$$

$$A_3 = \{a, b, d\}, \quad int(A_3) = \{a\}, \quad \bar{A}_3 = \{a, b, c, d, e, f\}, \quad \partial A_3 = \{b, c, d, e, f\}.$$

6 Basis $\mathcal{B}$ for a topological space $(X, \mathcal{O})$.

Definition: Let $(X, \mathcal{O})$ be a topological space. A collection $\mathcal{B}$ of open subsets of X is said to be a basis for the topology $\mathcal{O}$ if every open set in the space is a union of members of $\mathcal{B}$.

Thus, a basis $\mathcal{B}$ for a topology on X satisfies the following two properties.

1. Every point $x \in X$ lies in some set $B \in \mathcal{B}$.

2. For each pair of sets B_1, B_2 in $\mathcal{B}$ and each point $x \in B_1 \cap B_2$, there exists a set B_3 in $\mathcal{B}$ where $x \in B_3 \subset B_1 \cap B_2$.

The following examples are bases $\mathcal{B}_i$ of a given a topology $\mathcal{O}$ and set X.

For a topology $\mathcal{O} = \{X, \emptyset, \{a, b\}, \{a\}, \{b\}\}$, on a set $X = \{a, b\}$. Consider two bases

$$\mathcal{B}_1 = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}, \quad \mathcal{B}_2 = \{\emptyset, \{a\}, \{b\}\}.$$

For another topology, with $X = \mathbb{R}$, let the topology $\mathcal{O}$ be defined by the basis

$$\mathcal{B}_3 = \{(a, b) \mid a, b \in \mathbb{R} \quad and \quad a < b\}$$

where $(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$. We may conclude that $\mathcal{B}_3$ is a basis for a topology $\mathcal{O}$ on the set X, since for every $x \in R$ there exists an interval $(x - 2, x + 2)$ in $\mathcal{B}_3$ which contains x. This means that for every point x in the real numbers, x is contained in the set $(x - 2, x + 2)$, which is a set B found in the basis $\mathcal{B}_3$.

Next, considering an arbitrary pair of sets in $\mathcal{B}_3$, $B_1 = (a, b)$ and $B_3 = (c, d)$ which overlap, note an arbitrary point $x \in B_1 \cap B_2$. There exists an interval $B_3 = (max\{a, c\}, min\{b, d\})$ such that $x \in B_3 \subseteq B_1 \cap B_2$. Another basis, $\mathcal{B}_4$ for constructing a topology on the set $X = \mathbb{R}$ may be

$$\mathcal{B}_4 = \{(r, s) \mid r, s \in \mathbb{Q} \quad and \quad r < s\}.$$

where $(r, s) = \{x \in \mathbb{R} \mid r < x < s\}$.We may conclude that $\mathcal{B}_4$ is a basis for a topology $\mathcal{O}$ on the set X, since for every $x \in \mathbb{R}$ there exists an interval (r, s) in $\mathcal{B}_4$ which contains x. This means that for every point x in the real numbers, x is contained in the set (r, s), which is a set B found in the basis $\mathcal{B}_4$.

Next, considering an arbitrary pair of sets B in $\mathcal{B}_4$, $B_1 = (r_1, s_1)$ and $B_3 = (r_2, s_2)$ which overlap, note an arbitrary point $x \in B_1 \cap B_2$. There exists an interval $B_3 = (max\{r_1, r_2\}, min\{s_1, s_2\})$ such that $x \in B_4 \subseteq B_1 \cap B_2$.

7 Functions as metrics on a set X

Let X be a non-empty set and d a real-valued function defined on $X \times X$ such that for $a, b \in X$:

1. $d(a, b) \geq 0$, and $d(a, b) = 0$ if and only if $a = b$.

2. $d(a, b) = d(b, a)$

3. $d(a, c) \leq d(a, b) + d(b, c)$ for all a, b and c in X.

Thus, d is said to be a metric on X, and (X, d) is denoted as a metric space. Also, $d(a, b)$ is recognized as the distance between a and b.

Next, let d_i denote a function $\{(x, y) \mid x, y \in X_i\} \longrightarrow \mathbb{R}$ that is a metric on a set X_i. Here are some examples.

For $X = \mathbb{R}$, let the function $d_1\colon \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ be defined by

$$d_1(a, b) = \begin{cases} 0 \quad if \quad a = b \\ 1 \quad if \quad a \neq b \end{cases}$$

For $X = \mathbb{R}^2$, let the function $d_2\colon \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ be defined by

$$d_2 = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2}$$

For $X = \mathbb{R}$, let the function $d_3\colon \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ be defined by,

$$d_3 = \mid a - b \mid .$$

such that $a, b \in \mathbb{R}$.

8 Open balls on metric spaces

Let us define the set of open balls $B_r(x_j)$ on a metric space d_i for a given set X_i as

$$B_r(x_j) = \{y \mid y \in X \quad and \quad d(x, y) < r\}.$$

The open balls have radius r and are centered at x. Here are some examples.

1. For $X = \mathbb{R}$, let the function $d_1\colon \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ be defined by

$$d_1(a, b) = \begin{cases} 0 \quad if \quad a = b \\ 1 \quad if \quad a \neq b \end{cases}$$

Three open balls on this metric can be expressed as:

1. $B_2(1) = \{y \mid y \in \mathbb{R} \quad and \quad d_1(1, y) < 2\}$

2. $B_5(2) = \{y \mid y \in \mathbb{R} \quad and \quad d_1(2, y) < 5\}$

3. $B_{\frac{1}{2}}(0) = \{y \mid y \in \mathbb{R} \quad and \quad d_1(0, y) < \frac{1}{2}\}$

2. For $X = \mathbb{R}^2$, let the function d_2: $\mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ be defined by

$$d_2(\langle a_1, b_1\rangle, \langle a_1, b_2\rangle) = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2}$$

Three open balls on this metric can be expressed as:

1. $B_2(\langle 1, 3\rangle) = \{\langle y_1, y_2\rangle \mid \quad \langle y_1, y_2\rangle \in \mathbb{R}^2 \quad and \quad d_2(\langle 1, 3\rangle, \langle y_1, y_2\rangle) < 2\}$

2. $B_5(\langle 2, 4\rangle) = \{\langle y_1, y_2\rangle \mid \quad \langle y_1, y_2\rangle \in \mathbb{R}^2 \quad and \quad d_2(\langle 2, 4\rangle, \langle y_1, y_2\rangle) < 5\}$

3. $B_{\frac{1}{2}}(\langle 0, 6\rangle) = \{\langle y_1, y_2\rangle \mid \quad \langle y_1, y_2\rangle \in \mathbb{R}^2 \quad and \quad d_2(\langle 0, 6\rangle, \langle y_1, y_2\rangle) < \frac{1}{2}\}$

3. For $X = \mathbb{R}$, let the function d_3: $\mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ be defined by,

$$d_3(a, b) =\mid a - b \mid .$$

such that $a, b \in \mathbb{R}$.

Three open balls on this metric can expressed as:

1. $B_3(4) = \{y \mid y \in \mathbb{R} \quad and \quad d_3(4, y) < 3\}$

2. $B_{\frac{7}{8}}(7) = \{y \mid y \in \mathbb{R} \quad and \quad d_3(7, y) < \frac{7}{8}\}$

3. $B_{10}(9) = \{y \mid y \in \mathbb{R} \quad and \quad d_3(9, y) < 10\}$

9 Collection of all open balls $B_r(x)$ as a basis

Let us consider the Euclidean metric on $\mathbb{R}$ as mentioned previously. That is, for $X = \mathbb{R}$, let the function $d : (a, b)$ from $\mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ be defined by,

$$d =\mid a - b \mid .$$

such that $a, b \in \mathbb{R}$.

The collection of all open balls with this metric may be considered a basis such that

$$\mathcal{B} = \{B_r(x) \mid x \in X, r \in \mathbb{R}, r > 0\}.$$

For the Euclidean metric on $\mathbb{R}$, $B_r(x)$ is the open interval $(x-r, x+r)$. Thus, We may conclude that $\mathcal{B}$ is a basis for a topology $\mathcal{O}$ on the set X, since for every $x \in \mathbb{R}$ there exists an interval $(x-r, x+r)$ in $\mathcal{B}$ which contains x. This means that for every point x in the real numbers, x is contained in the set $(x-r, x+r)$, which is a set B found in the basis $\mathcal{B}$.

Next, considering an arbitrary pair of sets B in $\mathcal{B}$, $B_1 = (a-r, a+r)$ and $B_2 = (b-r, b+r)$ which overlap, note an arbitrary point $x \in B_1 \cap B_2$. There exists an interval $B_3 = (max\{a-r, b-r\}, min\{a+r, b+r\})$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

10 Continuous functions between topological spaces

Consider the definition of continuous function:

Definition: Let $(X, \mathcal{O}_x)$ and $(Y, \mathcal{O}_Y)$ be topological spaces and f a function from X into Y. Then $f : (X, \mathcal{O}_x) \longrightarrow (Y, \mathcal{O}_Y)$ is said to be a continuous mapping if for each $U \in \mathcal{O}_Y, f^{-1}(U) \in \mathcal{O}_x$.

Let us now consider three continuous functions between two topological spaces. Let us denote f_i as a continuous function between two topological spaces.

1. $f_1 : \mathbb{R} \longrightarrow \mathbb{R}$ such that

$$f(x) = x.$$

and the topology $\mathcal{O}_{\mathbb{R}}$ on $\mathbb{R}$ comes from the basis of open intervals in $\mathbb{R}$ such that

$$\mathcal{B}_{\mathbb{R}} = \{(m, M) \mid m < M\}.$$

2. $f_2 : \mathbb{R} \longrightarrow \mathbb{R}$ such that

$$f(x) = c.$$

and the topology $\mathcal{O}_{\mathbb{R}}$ on $\mathbb{R}$ comes from the basis of open intervals in $\mathbb{R}$ such that

$$\mathcal{B}_{\mathbb{R}} = \{(m, M) \mid m < M\}.$$

3. $f_3 : \mathbb{R} \longrightarrow Y$ such that

$$f(x) = \begin{cases} n \quad if \quad x < 0 \\ z \quad if \quad x = 0 \\ p \quad if \quad x > 0 \end{cases}$$

and $Y = \{p, n, z\}$, where the topology topology $\mathcal{O}_{\mathbb{R}}$ on $\mathbb{R}$ comes from the basis of open intervals in $\mathbb{R}$ such that

$$\mathcal{B}_{\mathbb{R}} = \{(m, M) \mid m < M\},$$

and the topology $\mathcal{O}_Y$ on Y is

$$\mathcal{O}_Y = \{\{p, n, z\}, \emptyset, \{p\}, \{n\}, \{p, n\}\}.$$

11 Demonstrating continuity of a function

Consider the function $f : \mathbb{R} \longrightarrow Y$ such that

$$f(x) = \begin{cases} n \quad if \quad x < 0 \\ z \quad if \quad x = 0 \\ p \quad if \quad x > 0 \end{cases}$$

and $Y = \{p, n, z\}$, where the topology $\mathcal{O}_{\mathbb{R}}$ on $\mathbb{R}$ comes from the basis of open intervals in $\mathbb{R}$ such that

$$\mathcal{B}_{\mathbb{R}} = \{(m, M) \mid m < M\},$$

and the topology $\mathcal{O}_Y$ on Y is

$$\mathcal{O}_Y = \{\{p, n, z\}, \emptyset, \{p\}, \{n\}, \{p, n\}\}.$$

Lemma 1 in Hatcher's notes states that "A function $f : X \longrightarrow Y$ is continuous if and only if $f^{-1}(C)$ is closed in X for each closed set C in Y."

We have the closed sets in Y as

$$\{n, z\}, \{z\}, \{p, z\}.$$

These sets are closed since their compliments, $X - U_i$, are open in $\mathcal{O}_Y$ as shown:

1. $Y \backslash \{n, z\} = \{p\}$

2. $Y \backslash \{z\} = \{p, n\}$

3. $Y \backslash \{p, z\} = \{n\}$

We then have the pre-images of these closed sets in Y.

1. $f^{-1}(\{n, z\}) = (-\infty, 0]$

2. $f^{-1}(\{z\}) = [0]$

3. $f^{-1}(\{p, z\}) = [0, \infty)$

They are all closed in X. Lemma 1 is satisfied.

Lemma 2 in Hatcher's notes states that"given a function $f : X \longrightarrow Y$ and a basis $\mathcal{B}$ for Y, then f is continuous if and only if $f^{-1}(B)$ is open in X for each $B \in \mathcal{B}$."

The topology $\mathcal{O}_Y$ on Y defined as

$$\mathcal{O}_Y = \{\{p, n, z\}, \emptyset, \{p\}, \{n\}, \{p, n\}\}.$$

is also a basis $\mathcal{B}_Y$. Thus, we must show that the pre-image of each subset B of $\mathcal{B}_Y$ is open in X.

We have the pre-image of each subset B in $\mathcal{B}_Y$.

1. $f^{-1}(\{p, , n, z\}) = \mathbb{R}$

2. $f^{-1}(\emptyset) = \emptyset$

3. $f^{-1}(\{p\}) = (0, \infty)$

4. $f^{-1}(\{n\}) = (-\infty, 0)$

5. $f^{-1}(\{p, n\}) = (-\infty, 0) \cup (0, \infty)$

As shown, the pre-image of each each set B in $\mathcal{B}_Y$ is open in X. Thus, the function is continuous.

12 Homeomorphism between distinct sets

Let us consider the definition of a homeomorphism.

Definition: Let $(X, \mathcal{O}_x)$ and $(Y, \mathcal{O}_Y)$ be topological spaces. Then they are homeomorphic if there exists a function $f : X \longrightarrow Y$ which has the following properties:

1. f is injective; $f(x_1) = f(x_2)$ implies $x_1 = x_2$.
2. f is surjective; for any $y \in Y$, there exists an $x \in X$ such that $f(x) = y$.
3. For each $U \in \mathcal{O}_Y$, $f^{-1}(U) \in \mathcal{O}_x$.
4. For each $V \in \mathcal{O}_x$, $f(V) \in \mathcal{O}_Y$.

The follwing is an example of a homeomorphism between the sets

$$X = \{a, b, c, d, e\}, \quad Y = \{g, h, i, j, k\}$$

with respective topologies

$$\mathcal{O}_X = \{X, \emptyset, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e\}\}, \quad \mathcal{O}_Y = \{Y, \emptyset, \{g\}, \{i, j\}, \{g, i, j\}, \{h, i, j, k\}\}.$$

Let the homeomorphism be defined by the function $f : X \longrightarrow Y$ such that

$$f = \{a \longmapsto g, b \longmapsto h, c \longmapsto i, d \longmapsto j, e \longmapsto k\}.$$